Complex Variables Notes

William Farmer

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Contents

1 Fundamentals and Techniques

1.1 Complex Numbers and Elementary Functions

1.1.1 Properties

We define an imaginary number as

$$i^2 = -1$$

While a complex number is defined as

$$z = x + iy$$

The common functions $\Re$ and $\Im$ yield the real and imaginary parts of a complex number respectively.[1] We can also express complex numbers in polar coordinates.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Using Euler's Identity,

$$\cos\theta + i\sin\theta = e^{i\theta}$$

the alternate form is defined as

$$\begin{aligned} z &= x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} \\ r &= \sqrt{x^2+y^2} = |z| \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

The complex conjugate is defined as

$$x - iy \equiv re^{-i\theta}$$

We can define some common equivalences.

- $\exp(2\pi i) = 1$
- $\exp(\pi i) = -1$
- $\exp\left(\frac{\pi i}{2}\right) = i$
- $\exp\left(\frac{3\pi i}{2}\right) = -i$
- $\exp(i\theta_1)\exp(i\theta_2) = \exp(i(\theta_1+\theta_2))$
- $\exp(i\theta)^m = \exp(im\theta)$
- $\exp(i\theta)^{1/n} = \exp\left(\frac{i\theta}{n}\right)$

Another neat trick is to let $z = 1/t$ to analyze behavior at ∞.

[1] Also denoted as Re and Im

1.1.2 Stereographic Projection

We can visualize complex numbers with a stereographic projection. Zero is located at the North Pole, and infinity at the South Pole.

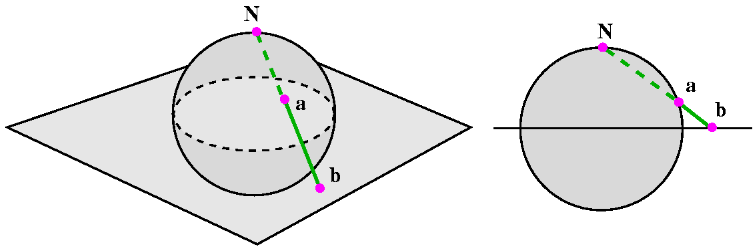

Figure 1: Stereographic Projection

These points are

$$X = \frac{4x}{|z|^2 + 4} \qquad Y = \frac{4y}{|z|^2 + 4} \qquad Z = \frac{2|z|^2}{|z|^2 + 4}$$

1.1.3 Elementary Functions

Similar to Real Analysis, we can define a neighborhood of some point z as the region enclosed by

$$|z - z_0| < \epsilon$$

As with sets, these can be closed, bounded, regions, domains, etc. . . .

We can also define functions of complex numbers, and as with real valued numbers, they mostly work the same. The simplest function is the power function.

$$f(z) = z^n$$

Which can be extended to define Polynomials and rational functions (as the result of dividing a polynomial function with another).

Limits also work the same, even with Radii of Convergence, etc.

Projections and Mappings work intuitively.

1.1.4 Example

Solve for all roots of the following equation: $z^4 + 2z = 0$.

$z\left(z^3 + 2\right) = 0$, so $z = 0$ or $z^3 = -2$, and then $r^3 = 2$, $e^{3i\theta} = e^{i\pi} \Rightarrow \theta = \pi/3 + 2\pi n/3$, $n = 0, 1, 2$. Thus, the roots are

$$z = 0, 2^{1/3}e^{i\pi/3}, 2^{1/3}e^{i\pi} = -2^{1/3}, 2^{1/3}e^{5i\pi/3}$$

1.1.5 Limits

Theorem 1 ($\epsilon - \delta$ Limit Definition)**.** *A complex limit can be defined as*

$$\lim_{z \to z_0} f(z) = w_0$$

if for every sufficiently small $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \qquad |z - z_0| < \delta$$

This is the traditional $\epsilon - \delta$ format that we're used to from real analysis.

Similarly, a function is said to be continuous if for all z,

$$\lim_{z \to z_0} f(z) = z_0$$

The traditional definitions of Uniform and Absolute convergence also apply.

Using these limit definitions we can define the concept of a derivative.

$$f'(z_0) = \lim_{\Delta z \to 0} \left(\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$$

1.2 Analytic Functions and Integration

1.2.1 Analytic Functions

In order for a complex function to be differentiable, it has to satisfy the Cauchy-Riemann Conditions.

Theorem 2 (Cauchy-Riemann Conditions)**.** *By writing the real and imaginary parts separately in the definition of a derivative, we get*

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ f'(z) &= \lim_{\Delta x \to 0} \left(\frac{u(x + \Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x,y)}{\Delta x} \right) \\ &= u_x(x,y) + iv_x(x,y) \end{aligned}$$

Yielding the Cauchy-Riemann conditions,

$$\begin{aligned} u_x &= v_y \qquad & v_x &= -u_y \\ u_r &= \frac{v_\theta}{r} \qquad & v_r &= -\frac{u_\theta}{r} \end{aligned}$$

Theorem 3. *The function $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z = x + iy$ of a region in the complex plane if and only if the partial derivatives u_x, u_y, v_x, v_y, are continuous and satisfy the Cauchy-Riemann conditions at $z = x + iy$.*

For differentiability, we can use the term analyticity to mean the same thing, both for pointwise differentiability and differentiability over a region. Points that are not differentiable (analytic) are called singular points.[2]

Some properties follow.[3]

- Sums, Products, and Compositions of analytic functions are analytic.
- The reciprocal of an analytic function that is nowhere zero is analytic, as is the inverse of an invertible analytic function whose derivative is nowhere zero.

[2]Holomorphic is sometimes used as well (or instead) of analytic.

[3]`https://en.wikipedia.org/wiki/Analytic_function#Properties_of_analytic_functions`

An entire function is one that's analytic on the entire finite plane.

Taking the second derivative of the Cauchy-Riemann conditions yields Laplace's Equation.

$$u_{xx} = v_{xy} \qquad v_{yx} = -u_{yy}$$
$$\nabla^2 w = 0 \Rightarrow \begin{cases} \nabla^2 u \equiv u_{xx} + u_{yy} = 0 \\ \nabla^2 v \equiv v_{xx} + v_{yy} = 0 \end{cases}$$

A function that satisfies the concise Laplace Equation: $\nabla^2 w = 0$ is called a harmonic function in D. u and v are referred to as harmonic functions in D, and they are harmonic conjugates of each other.

1.2.2 Example

Let $f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$. Verify Cauchy-Riemann for all x, y, and then show that $f'(z) = e^z$.

$$\begin{aligned} u = e^x \cos y \qquad v = e^x \sin y \\ u_x = e^x \cos y = v_y \\ v_y = -e^x \sin y = -v_x \\ f'(z) = u_x + iv_x = e^x(\cos y + i \sin y) = e^z \end{aligned}$$

1.2.3 Ideal Fluid Flow - Application of Laplace's Equation

Two dimensional ideal fluid flow is a great example of Laplace's Equation. This is fluid that is time independent, nonviscous, incompressible, and irrotational.

1. Incompressibility:

$$v_{1,x} + v_{2,y} = 0$$

   Where v_1 and v_2 are the horizontal and vertical components.

2. Irrotationality:

$$v_{2,x} - v_{1,y} = 0$$

3. Simplified:

$$\begin{aligned} v_1 = \phi_x = \psi_y \qquad v_2 = \phi_y = -\psi_x \\ \mathbf{v} = \nabla \phi \end{aligned}$$

   ϕ is the velocity potential, and ψ the stream function. Cauchy-Riemann is satisfied for ϕ and ψ, therefore we have a complex velocity potential.

$$\begin{aligned} \Omega(z) &= \phi(x, y) + i\psi(x, y) \\ \Omega'(z) &= \phi_x + i\psi_x = \phi_x - i\psi_y = v_1 - v_2 \end{aligned}$$

1.2.4 Example

Uniform Flow is

$$\Omega(z) = v_0 e^{-i\theta_0} z = v_0(\cos\theta_0 - i\sin\theta_0)(x + iy)$$

where v_0 and θ_0 are positive real constants. The corresponding velocity potential and velocity field is given by

$$\phi(x, y) = v_0(\cos(\theta_0 x) + \sin(\theta_0 y))$$
$$v_1 = \phi_x = v_0\cos\theta_0 \qquad v_2 = \phi_y = v_0\sin\theta_0$$

which is identified with uniform flow making an angle θ_0 with the x axis. Alternatively, the steam function $\psi(x, y) = v_0(\cos(\theta_0 y) - \sin(\theta_0 x)) = \text{const.}$ reveals the same flow field.

1.2.5 Multivalued Functions

A simple example of this is the square root function which takes on different values for n even or odd.

$$z = w^2 \qquad w = \sqrt{z}$$
$$= r^{1/2} e^{i\theta_p/2} e^{n\pi i}$$

We can define these "points" where complex functions take on multiple values as branch points. In the same way that they're referred to as branch points, branches of a multivalued function are when we restrict to only one set of continuous values. A branch cut is this restriction process.[4]

Log is more complicated, and we define it as such.

$$w = \log(z) = \log r + i\theta_p + 2n\pi i, \qquad n = 0, \pm1, \pm2, \ldots, \qquad 0 \le \theta_p < 2\pi$$

1.2.6 Example

Find the location of the branch points and discuss possible branch cuts for the following functions:

1. $(z - i)^{1/3}$
   Let $z - i = \epsilon e^{i\theta_p}$ which is a circular contour centered at $z = i$. We have just a power function in terms of $\zeta = z - i$, so $z = i$ and $z = \infty$ are branch points. Any line connecting $z = \infty$ and $z = i$ is a branch cut, e.g. $\{z = iy | y \in [1, +\infty)\}$ is as good as any. There are 3 distinct branches.

2. $\log\left(\frac{1}{z-2}\right)$
   $\log\left(\frac{1}{z-2}\right) = -\log(z - 2)$. Again, this is $-\log(z)$ but with shiftd origin. So the branch points are $z = 2$ and $z = \infty$. A branch cut must connect the branch points, it can be $\{z = x | x \in [2, +\infty)\}$ or $\{z = x | x \in (-\infty, 2]\}$.

[4] The real analogy here is a function like $\pm\sqrt{x}, x \in \mathbb{R}$. 0 is a branch point, and we often times just examine the branch where $\sqrt{x} > 0$. The analogous branch cut is $x > 0$.

1.2.7 Example

Solve for all values of z: $4 + 2e^{z+i} = 2$.

$$4 + 2e^{z+i} = 2 \Rightarrow e^{z+i} = -1 = e^{i\pi + 2\pi i n}, n \in \mathbb{Z}$$

Therefore

$$z + i = i\pi + 2\pi i n \Rightarrow z = i(\pi - 1 + 2\pi n), n \in \mathbb{Z}$$

1.2.8 Example

Find the location of the branch points and discuss a branch cut structure associated with the function:

- $f(z) = \frac{z-1}{z}$
  This is a rational function singular at $z = 0$, but single-valued, so no branch points.
- $f(z) = \log\left(z^2 - 3\right)$
  Here $z^2 - 3$ is entire single-valued function so the only branch points are those where $z^2 - 3 = 0$ or $z^2 - 3 = \infty$. Thus, there are three branch points, $z = \pm\sqrt{3}$, and $z = \infty$. A branch cut must make sure there is no possibility going around and single of them, in this case it must connect all three points. E.g. consider a cut on real axis $\{z = x | x \in [-3, +\infty)\}$.
- $f(z) = \exp\sqrt{z^2 - 1}$
  Since function e^z is entire (analytic on plane) the only possible branch points are those of $\sqrt{z^2 - 1}$, i.e. $z = \pm 1$ and $z = \infty$. However, doing the circle argument $z - 1 = r_1 e^{i\theta_1}$, $z + 1 = r_2 e^{i\theta_2}$, $\theta_1 \to \theta_1 + 2\pi$, $\theta_2 \to \theta_2 + 2\pi$, one sees that $z = \infty$ is not a branch point since $\exp(2\pi i + 2\pi i)/2 = 1$ which corresponds to encircling both $z = 1$ and $z = -1$, equivalent to encircling just $z = \infty$. Thus, $z = \infty$ is not a branch point even for $\sqrt{z^2 - 1}$. But $z = \pm 1$ are branch points, and a branch cut connecting them is $\{z = x | x \in [-1, 1]\}$.

1.2.9 More Complicated Multivalued Functions and Riemann Surfaces

If we have functions like the following

$$w = [(z - a)(z - b)]^{1/2}$$

We need to use a slightly more complicated branch cut/structure. We know that the points $z = a, b$ are both branch points (by letting $z = a + \epsilon_1 e^{i\theta_1}$ and as θ_1 varies from 0 to 2π, w jumps from $q^{1/2}$ to $-q^{1/2}$), and so we can define a branch cut as follows.

$$\begin{aligned} z - b &= r_1 e^{i\theta_1} \\ z - a &= r_2 e^{i\theta_2} \qquad 0 \le \theta_1, \theta_2 < 2\pi \end{aligned}$$

Our equation now becomes

$$w = (r_1 r_2)^{1/2} e^{i(\theta_1 + \theta_2)/2}$$

This process extends to more complicated functions, as for any w of the form

$$w = [(z - x_1)(z - x_2) \cdots (z - x_n)]^m$$

we can define our branch cuts to be

$$z - x_k = r_k e^{i\theta_k}$$

yielding

$$w = (r_1 r_2 \cdots r_n) e^{mi(\theta_1 + \theta_2 + \cdots + \theta_n)}$$

1.2.10 Example

Find the location of branch points and discuss a branch cut structure associated with the function:

$$f(z) = \coth^{-1} \frac{z}{a} = \frac{1}{2} \log \left(\frac{z+a}{z-a} \right), a > 0$$

This is (up to a constant) log of rational function, so the branch points are those where $(z+a)/(z-a) = 0$ or ∞, i.e. $z = \pm a$. As for $z = \infty$, it is not a branch point, as the limit equals 1, not zero. A cut must connect the two points, so a possible one is interval $[-a, a]$ on the real axis.

1.2.11 Riemann Surfaces

Instead of considering the normal complex plane with arbitrary "cuts", it can be useful to instead consider a surface with multiple "sheets". Any multivalued function only has one point that corresponds to each point on the sheet. This way, for any given sheet, the function is single-valued.

For the function $w^{1/2}$, since we have two branches, our Riemann surface is two-sheeted. For the log function, since it is infinitely multivalued, we have infinite sheets.

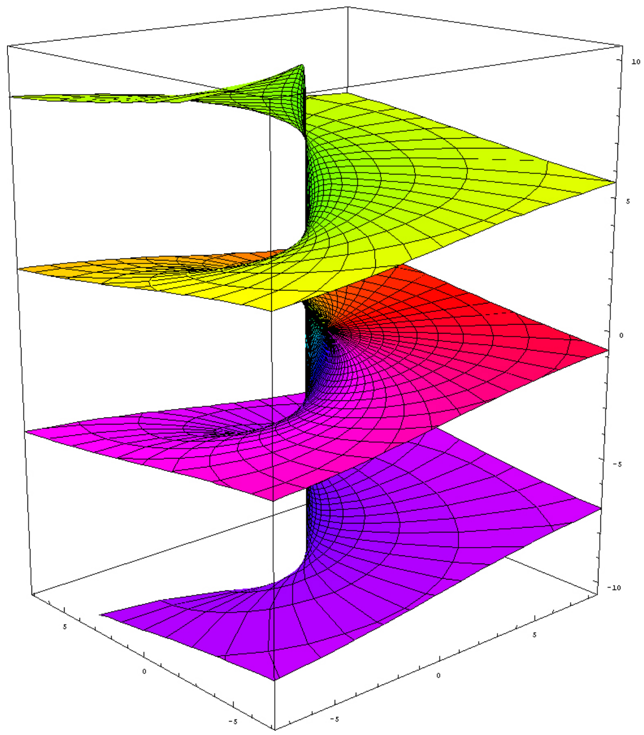

Figure 2: Riemann Surface for $\log(z)$

1.2.12 Complex Integration

Consider a function $f(t) = u(t) + iv(t)$. This function is integrable if u and v are integrable (with the same properties applying).

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Defining a curve on the complex plane can be done parametrically, with form[5]

$$z(t) = x(t) + iy(t)$$

The path (contour) integral of function f on contour z is defined to be[6]

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$$

This is really a line integral in the (x, y) plane.

[5] These curves are

- **Simple Curve** or **Jordan Arc** if it does not intersect itself.
- **Simple Closed Curve** or **Jordan Curve** if the endpoints meet.

[6] Contours are defined as piecewise smooth connected arcs. Simple closed is referred to as a Jordan Contour

Theorem 4. *Suppose $F(z)$ is an analytic function and that $f(z) = F'(z)$ is continuous in a domain D. Then for a contour C lying in D with endpoints z_1 and z_2*

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

Since we can think of the parameterized complex plane as a vector field, for closed curves, we have

$$\oint_C f(z)\,dz = \oint_C F'(z)\,dz = 0$$

Note that everything here hinges on the analyticity of F and the continuity in domain D.

Theorem 5. *Let $f(z)$ be continuous on a contour C. Then*

$$\left|\int_C f(z)\,dz\right| \le ML$$

where L is the length of C and M is an upper bound for $|f|$ on C.

Arc length can be defined (from Calc III) for a parameterized curve with form $z(t) = u(t) + iv(t)$ as

$$\int_a^b \sqrt{(u'(t))^2 + (v'(t))^2}\,dt$$

1.2.13 Example

Evaluate $\int_C \overline{z}\,dz$ for a contour from $z = 0$ to $z = 1$ to $z = 1 + i$.

$$\begin{aligned}
\int_C \overline{z}\,dz &= \int_C (x - iy)(dx + i\,dy) \\
&= \int_{x=0}^1 x\,dx + \int_{y=0}^1 (1 - iy)(i\,dy) \\
&= \frac{1}{2} + i\big[y - iy^2/2\big]_0^1 \\
&= 1 + i
\end{aligned}$$

1.2.14 Cauchy's Theorem

Theorem 6 (Cauchy)**.** *If a function f is analytic in a simply connected domain D, then along a simple closed contour C in D*

$$\oint_C f(z)\,dz = 0$$

We also require that $f'(z)$ is also continuous in D.

"If $f(z)$ is analytic everwhere interior to and on a simple closed contour C, then $\oint_C f(z)\,dz = 0$."

Again, NOTE that everything hinges on the fact that D must be simply connected. In order to use this, you need a simply connected domain D AND a simple closed contour C.

To best apply Cauchy's Theorem, we can use tricks like turning a complex contour into several simple contours, and deforming a simply connected domain so that the function is analytic on the domain.

1.2.15 Cauchy's Integral Formula, Its $\overline{\partial}$ Generalization and Consequences

Theorem 7. *Let $f(z)$ be analytic interior to and on a simple closed contour C. Then at any interior point z*

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

This is referred to as Cauchy's Integral Formula.

Theorem 8. *If $f(z)$ is analytic interior to and on a simple closed contour C then all the derivatives $f^{(k)}(z), k = 1, 2, \ldots$ exist in the domain D interior to C, and*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$

Theorem 9. *All partial derivatives of u and v are continuous at any point where $f = u + iv$ is analytic.*

Theorem 10 (Lioville)**.** *If $f(z)$ is entire and bounded in the z plane (including infinity), then $f(z)$ is a constant.*

Theorem 11 (Morera)**.** *If $f(z)$ is continuous in a domain D and if*

$$\oint_C f(z)\, dz = 0$$

for every simple closed contour C lying in D, then $f(z)$ is analytic in D.

Theorem 12 (Maximum Principles)**.** *1. If $f(z)$ is analytic in a domain D, then $|f(z)|$ cannot have a maximum in D unless $f(z)$ is a constant.*

2. If $f(z)$ is analytic in a bounded region D and $|f(z)|$ is continuous in the closed region $\overline{D}$, then $|f(z)|$ assumes its maximum on the boundary of the region.

Theorem 13 (Generalized Cauchy Formula)**.** *If $\partial f/\partial\overline{\zeta}$ exists and is continuous in a region R bounded by a simple closed contour C, then at any interior point z*

$$f(z) = \frac{1}{2\pi i} \oint_C \left(\frac{f(\zeta)}{\zeta - z}\right) d\zeta - \frac{1}{\pi} \iint_R \left(\frac{\partial f/\partial\overline{\zeta}}{\zeta - z}\right) dA(\zeta)$$

1.2.16 Theoretical Developments

Theorem 14 (Cauchy-Goursat)**.** *If a function $f(z)$ is analytic at all points interior to and on a simple closed contour, then*

$$\oint_C f(z)\, dz = 0$$

1.3 Sequences, Series, and Singularities of Complex Functions

1.3.1 Definitions of Complex Sequences, Series, and Their Basic Properties

We can denote a sequence of functions that converge to some given function as

$$\lim_{n\to\infty} f_n(z) = f(z) \Leftrightarrow |f_n(z) - f(z)| < \epsilon$$

If the limit does not exist, or is infinite, the sequence is said to diverge for those values of z.

We say the sequence of functions converges uniformly if we can choose N on only ϵ, and not z. In other words, if for any z, the n^{th} function is ϵ close to $f(z)$.

Theorem 15. *Let the sequence of functions $f_n(z)$ be continuous for each integer n and let $f_n(z)$ converge to $f(z)$ uniformly in a region $\mathcal{R}$. Then $f(z)$ is continuous, and for any finite contour C inside $\mathcal{R}$*

$$\lim_{n\to\infty}\int_C f_n(z)\,dz = \int_C f(z)\,dz$$

Theorem 16 (Weierstrass M Test)**.** *Let $|b_j(z)| \le M_j$ in a region $\mathcal{R}$, with M_j constant. If $\sum_{j=1}^{\infty} M_j$ converges, then the series $S(z) = \sum_{j=1}^{\infty} b_j(z)$ converges uniformly in $\mathcal{R}$.*

Theorem 17 (Corollary: Ratio Test)**.** *Suppose $|b_1(z)|$ is bounded, and*

$$\left|\frac{b_{j+1}(z)}{b_j(z)}\right| \le M < 1, \qquad j > 1$$

for M constant. Then the series

$$S(z) = \sum_{j=1}^{\infty} b_j(z)$$

is uniformly convergent.

1.3.2 Taylor Series

A power series about the point $z = z_0$ is defined as

$$f(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

$$f(z + z_0) = \sum_{j=0}^{\infty} b_j z^j$$

With b_j, z_0 are constants. WLOG we can work with

$$f(z) = \sum_{j=0}^{\infty} b_j z^j$$

which is the $z_0 = 0$ case.

Theorem 18. *If the series*

$$f(z) = \sum_{j=0}^{\infty} b_j z^j$$

converges for some $z_* \neq 0$, then it converges for all z in $|z| < |Z_*|$. Morever, it converges uniformly in $|z| \le R$ for $R < |Z_*|$.

Theorem 19 (Taylor Series)**.** *Let $f(z)$ be analytic for $|z| \leq R$. Then*

$$f(z) = \sum_{j=0}^{\infty} b_j z^j$$

where

$$b_j = \frac{f^{(j)}(0)}{j!}$$

converges uniformly in $|z| \leq R_1 < R$.

The largest number R for which the power series converges inside the disk $|z| < R$ is called the radius of convergence.

Theorem 20. *Let $f(z)$ be analytic for $|z| \leq R$. Then the series obtained by differentiating the Taylor series termwise converges uniformly to $f'(z)$ in $|z| \leq R_1 < R$.*

Theorem 21. *If the power series converges for $|z| \leq R$, then it can be differentiated termwise to obtain a uniformly convergent series for $|z| \leq R_1 < R$.*

Theorem 22 (Comparison Test)**.** *Let the series $\sum_{j=0}^{\infty} a_j z^j$ converge for $|z| < R$. If $|b_j| \leq |a_j|$ for $j \geq J$, then the series $\sum_{j=0}^{\infty} b_j z^j$ also converges for $|z| < R$.*

Theorem 23. *Let each of two functions $f(z)$ and $g(z)$ be analytic in a common domain D. If $f(z)$ and $g(z)$ coincide in some subportion $D' \subset D$ or on a curve Γ interior to D, then $f(z) = g(z)$ everywhere in D.*

Theorem 24. *Let D_1 and D_2 be two disjoint domains, whose boundaries share a common contour Γ. Let $f(z)$ be analytic in D_1 and continuous in $D_1 \cup \Gamma$ and $g(z)$ be analytic in D_2 and continuous in $D_2 \cup \Gamma$, and let $f(z) = g(z)$ on Γ. Then the function*

$$H(z) = \begin{cases} f(z) & z \in D_1 \\ f(z) = g(z) & z \in \Gamma \\ g(z) & z \in D_2 \end{cases}$$

is analytic in $D = D_1 \cup \Gamma \cup D_2$. We say that $g(z)$ is the analytic continuation of $f(z)$.

Theorem 25. *If $f(z)$ is analytic and not identially zero in some domain D containing $z = z_0$, then its zeroes are isolated; that is, there is a neighborhood about $z = z_0$, $f(z_0) = 0$, in which $f(z)$ is nonzero.*

1.3.3 Laurent Series

Theorem 26 (Laurent Series)**.** *A function $f(z)$ analytic in an annulus $R_1 \leq |z - z_0| \leq R_2$ may be represented by the expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$

in the region $R_1 < R_a \leq |z - z_0| \leq R_b < R_2$, where

$$C_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

and C is any simple closed contour in the region of analyticity enclosing the inner boundary $|z - z_0| = R_1$.

Theorem 27. *The Laurent series defined above of a function $f(z)$ that is analytic in an annulus $R_1 \leq |z - z_0| \leq R_2$ converges uniformly to $f(z)$ for $\rho_1 \leq |z - z_0| \leq \rho_2$, where $R_1 < \rho_1$ and $R_2 > \rho_2$.*

Theorem 28. *Suppose $f(z)$ is represented by a uniformly convergent series*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$$

in the annulus $R_1 \leq |z - z_0| \leq R_2$. Then $b_n = C_n$, with C_n previously defined.